American Finance & Banking Review; Vol. 4, No. 2; 2019
ISSN 2576-1226 E-ISSN 2576-1234
Published by Centre for Research on Islamic Banking & Finance and Business, USA

# USA Income Distribution Counter-Business-Cyclical Trend

## (Estimating Lorenz curve using Continuous L1 norm estimation)

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**Abstract**
In this paper, the $L_1$ norm of continuous functions and corresponding continuous estimation of regression parameters are defined. The continuous $L_1$ norm estimation problems of linear one and two parameters models are solved. We proceed to use the functional form and parameters of the probability distribution function of income to exactly determine the $L_1$ norm approximation of the corresponding Lorenz curve of the statistical population under consideration. U.S. economic data used to estimate income distribution. An interesting finding of these calculations is that the distribution of income obeys counter-wise business cycles fluctuations. This finding is a new area for research in the realm of the theory and application of income distribution and business cycles interrelationship.

**Keywords:** Income Distribution, Lorenz Curve, L1 norm statistics, Business Cycle

**JEL:** C63

## 1. Introduction

The skewness of income distribution is persistently exhibited for different populations and at different times. It is discussed that Pearsonian family distributions are rival functions to explain income distribution. Lorenz curve is a method to analyze the skew distributions. There is a relation between the area under the Lorenz curve and the corresponding probability distribution function of the statistical population (see, Kendall and Stuart (1977)). That is, when the probability distribution function is known, we may find the corresponding Gini index as the measure of inequality.

Estimation of the Lorenz curve is confronted with some difficulties. For this estimation, we should define an appropriate functional form which can accept different curvatures (see, Bidabad and Bidabad (1989a,b)). There is another problem, that is, to create the necessary data set for estimating the corresponding parameters of the Lorenz curve, a large amount of computation on raw sample income data is inevitable. Obviously, these problems, despite their computational difficulties, make the significance of the estimated parameters poor (see, Bidabad and Bidabad (1989a,b)). To avoid this, we try to estimate the functional form of the Lorenz curve by using continuous information. In this paper, we use the probability density function of population income to estimate the Lorenz function parameters. The continuous $L_1$ norm smoothing method, which will be developed for estimating the regression parameters, is used to solve this problem. However, we concentrate on two rival probability density functions of Pareto and log-normal. Since the former is simply integrable, there is no general problem to derive the corresponding Lorenz function, and the function is uniquely derived. But in the latter case, the log-normal density function (which has better performance for full income range) than Pareto distribution (which better fits to higher income range, (see, Cramer (1973), Singh and Maddala (1976), Salem and Mount (1974)), is not integrable and we can not determine its corresponding Lorenz function. In this regard, we should solve the problem by defining a general Lorenz curve functional form and applying the $L_1$ norm smoothing to estimate the corresponding parameters.

In this paper, continuous $L_1$ norm estimation is developed by using a similar method proposed in Bidabad (1987a,88a,89a,b) for the discrete case. Then the method is applied to the estimation of the Lorenz curve functional forms which have been proposed by Gupta (1984) and Bidabad and Bidabad (1989,92). In the end, we use our formulation to estimate Gini index and Kakwani length indices of inequality for the United States for the period of 1971-1990, based on the assumption that income is distributed log-normally.

## 2. L1 norm of continuous functions

Generally, $L_p$ norm of a function f(x) (see, Rice and White (1964)) is defined by,

$$||f(x)||_p = \int_{x\in I} (|f(x)|^p dx)^{1/p} \qquad (1)$$



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Where, "I" is a closed bounded set. The $L_1$ norm of f(x) is simply written as,

$$||f(x)||_1 = \int_{x\varepsilon I} |f(x)|dx \quad (2)$$

Suppose that the non-stochastic function f(x,**β**) of "x", is combined with stochastic disturbance term "u" to form y(x) as follows,

$$y(x) = f(x, \boldsymbol{\beta}) + u \quad (3)$$

Where, **β** is unknown parameters vector. Rewriting u as the residual of y(x)-f(x,**β**), for $L_1$ norm approximation of "**β**" we should find "**β**" vector such that the $L_1$ norm of "u" is minimum. That is,

$$\underset{\boldsymbol{\beta}}{\text{Min:}}\ S=||u||_1=||y(x)-f(x,\boldsymbol{\beta})||_1=\int_{x\varepsilon I} |y(x)-f(x,\boldsymbol{\beta})|dx \quad (4)$$

## 3. Linear one parameter L1 norm continuous smoothing

Redefine f(x,**β**) as βx and y(x) as the following linear function,

$$y(x) = \beta x + u \quad (5)$$

Where, "**β**" is a single (non-vector) parameter. Expression (4) reduces to:

$$\underset{\beta}{\text{min:}}\ S = ||u||_1 = ||y(x)- \beta x||_1 = \int_{x\varepsilon I} |y(x)-f(x,\beta)|dx \quad (6)$$

The discrete analog of (6) is solved by Bidabad (1987a,88a,89a,b). In these papers, we proposed applying discrete and regular derivatives to the discrete problem by using a slack variable "t" as a point to distinguish negative and positive residuals. A similar approach is used here to minimize (6). To do so in this case, certain Lipschitz conditions are imposed on the functions involved (see, Usow (1967a)). Rewrite (6) as follows,

$$\underset{\beta}{\text{Min:}}\ S = \int_{x\varepsilon I} |x|\,|y(x)/x - \beta|dx \quad (7)$$

For convenience, define "I" as a closed interval [0,1]. The procedure may be applied to other intervals with no major problem (see, Usow (1967a), Hobby and Rice (1965), Kripke and Rivlin (1965)). To minimize this function, we should first remove the absolute value sign of the expression after the integral sign. Since "x" belongs to closed interval "I", y(x) (which is a linear function of "x") and also y(x)/x are smooth and continuous. Thus, since y(x)/x is uniformly increasing or decreasing function of "x", a value of tεI can be found to have the following properties,

$$\begin{aligned} y(x)/x &< \beta \quad \text{if } x < t \\ y(x)/x &= \beta \quad \text{if } x = t \\ y(x)/x &> \beta \quad \text{if } x > t \end{aligned} \quad (8)$$

Value of the slack variable "t" actually is the border of negative and positive residuals. If the value of "t" were known, from (8) (middle equation), we could calculate the optimal value of "β" or inversely. But nor "t" neither "β" are known. To solve this problem, according to (8), we can rewrite (7) as two separate definite integrals with different upper and lower bounds.

$$\underset{\beta}{\text{min:}}\ S = -\int_0^t |x|\,(y(x)/x - \beta)dx + \int_t^1 |x|\,(y(x)/x - \beta)dx \quad (9)$$

Decomposition of (7) into (8) has been done by use of the slack variable "t". Since both "β" and "t" are unknown, to solve (9), we partially differentiate it with respect to "t" and "β" variables.

$$\frac{\delta S}{\delta \beta} = \int_0^t |x|dx - \int_t^1 |x|dx = 0 \quad (10)$$

and using Liebniz' rule to differentiate the integrals with respect to their variable bounds "t", yields,

$$\frac{\delta S}{\delta t} = -|t|\,[\frac{y(t)}{t} - \beta] - |t|\,[\frac{y(t)}{t} - \beta] = 0 \quad (11)$$

Since "x" belongs to [0,1], equation (10) can be written as,

$$\int_0^t x dx - \int_t^1 x dx = 0 \quad (12)$$

or,

$$\tfrac{1}{2} t^2 - \tfrac{1}{2} + \tfrac{1}{2}t^2 = 0 \quad (13)$$

Which yields,

$$t = \sqrt{2}/2 \quad (14)$$

Substitute for "t" in equation (11), yields,



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$$\beta = \frac{y(\surd 2/2)}{\surd 2/2} \tag{15}$$

Remember that y(t) is function y(x) evaluated at x=t. Value of "β" given by (15) is the optimal solution of (6). The above procedure actually is a generalization of Laplace weighted median for the continuous case.

Before applying this procedure to the Lorenz curve, let us develop the procedure for the two parameters linear model.

## 4. Linear two parameters L1 norm continuous smoothing

Now, we try to apply the above technique to the linear two parameters model. Rewrite (4) as,

$$\underset{\alpha,\beta}{\text{Min:}}\ S=||u||_1=||y(x)-\alpha-\beta x||_1=\int_{x\varepsilon I}|y(x)-\alpha-\beta x|dx \tag{16}$$

Where, "α" and "β" are two single (non-vector) unknown parameters and y(x) and "x" are as before. According to Rice (1964c), let f(α*,β*,x) interpolates y(x) at the set of canonical points $\{x_i;i=1,2\}$, if y(x) is such that y(x)-f(α*,β*,x) changes sign at these $x_i$'s and at no other points in [0,1], then f(α*,β*,x) is the best $L_1$ norm approximation to y(x) (see also, Usow (1967a)). With the help of this rule, if we denote these two points to $t_1$ and $t_2$ we can rewrite (16) for I=[0,1] as,

$$S = \int_0^{t_1} [y(x)-\alpha-\beta x]dx - \int_{t_1}^{t_2} [y(x)-\alpha-\beta x]dx + \int_{t_2}^{1} [y(x)-\alpha-\beta x]dx \tag{17}$$

Since $t_1$ and $t_2$ are also unknowns, we should minimize S with respect to α, β, $t_1$ and $t_2$. Taking partial derivative of (17) using Liebniz' rule with respect to these variables and equating them to zero, we will have,

$$\frac{\delta S}{\delta \alpha} = - \int_0^{t_1} dx + \int_{t_1}^{t_2} dx - \int_{t_2}^{t_1} dx = 0 \tag{18}$$

$$\frac{\delta S}{\delta \beta} = - \int_0^{t_1} dx + \int_{t_1}^{t_2} dx - \int_{t_2}^{t_1} dx = 0 \tag{19}$$

$$\frac{\delta S}{\delta t_1} = 2[y(t_1) -\alpha-\beta t_1] = 0 \tag{20}$$

$$\frac{\delta S}{\delta t_2} = - 2[y(t_2) -\alpha - \beta t_2] = 0 \tag{21}$$

Equations (18) through (21) may be solved simultaneously for α, β, $t_1$ and $t_2$. Thus, we have the following system of equations,

$$2t_2 - 2t_1 - 1 = 0 \tag{22}$$

$$t_2^2 - t_1^2 - ½ = 0 \tag{23}$$

$$y(t_1) - \alpha - \beta t_1 = 0 \tag{24}$$

$$y(t_2) - \alpha - \beta t_2 = 0 \tag{25}$$

The solutions are,

$$t_1=1/4 \tag{26}$$

$$t_2=3/4 \tag{27}$$

$$\alpha = y(3/4)-(3/4)\beta = y(1/4)-(1/4)\beta \tag{28}$$

$$\beta = 2[y(3/4)-y(1/4)] \tag{29}$$

This procedure, similar to that of multiple regression model for discrete case may be expanded to include "m" unknown parameters which is not discussed here. Some computational methods for solving the different cases of m parameters model are investigated by Ptak (1958), Rice and White (1964), Rice (1964a,b,c,69,85), Usow (1967a), Lazarski (1975a,b,c,77) (see also, Hobby and Rice (1965), Kripke and Rivlin (1965), Watson (1981)). Now, let us have a look at Lorenz curve and its proposed functional forms.

## 5. Lorenz curve

The Lorenz curve for a random variable with probability density function f(v) may be defined as the ordered pair[1],

$$\left(P(V|V \le v), \frac{E(V|V \le v)}{}\right) \qquad v\varepsilon R \tag{30}$$

[1] Taguchi (1972a,b,c,73,81,83,87,88) multiplies the second element of (30) by P(V|V≤v) which is not correct; his definition of (31) is equivalent to ours.



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E(V)

Where "P" and "E" stand for probability and expected value operators. For a continuous density function f(v), (30) can be written as,

$$\left(\int_{-\infty}^{v} f(w)dw, \frac{\int_{-\infty}^{v} wf(w)dw}{\int_{-\infty}^{+\infty} wf(w)dw}\right) \equiv (x(v),y(x(v))) \quad (31)$$

We denote (31) by (x(v),y(x(v))) where x(v) and y(x(v)) are its elements. Therefore, "x" is a function which maps "v" to x(v) and "y" is a function which maps x(v) to y(x(v)). The function y(x(v)) is simply the Lorenz curve function. In recent years some functional forms for the Lorenz curve have been introduced. Among different proposed functions, we use the forms of Gupta (1984) and Bidabad and Bidabad (1989,92) which benefits from certain properties (see the papers for more explanations). Gupta (1984) proposed the functional form,

$$y=xA^{x-1} \qquad A>1 \quad (32)$$

Bidabad and Bidabad (1989,92) suggest the following functional form:

$$y=x^{B}A^{x-1} \qquad B\geq 1,\ A\geq 1 \quad (33)$$

To estimate the above functions by regular estimating method, we should gather discrete data from the statistical population, and manipulate them to construct relevant x and y vectors to estimate "A" of (32) or "A" and "B" of (33). If the probability distribution of income is known, instead of gathering discrete observations, we can estimate the Lorenz curve by using the continuous $L_1$ norm smoothing method for continuous functions. In the following section, we proceed to apply this method to estimate the parameters "A" of (32) and "A" and "B" of (33) by using the information of probability density function of income.

## 6. Continuous L1 norm smoothing of Lorenz curve

To estimate the Lorenz curve parameters when income probability density function is known, we cannot always take straightforward steps. When the probability density function is easily integrable, there is no major problem in advance. We can find the functional relationship between the two elements of (31) by simple mathematical derivation. But, when integrals of (31) are not obtainable, another procedure should be adopted.

Suppose that income of a society is distributed with probability density function f(w). This density function may be a skewed function such as Pareto or log-normal, as follows

$$f(w)=\theta k^{\theta}w^{-\theta-1}, \qquad w,k>0,\ \theta>0 \quad (34)$$

$$f(w)=[1/w\sigma\sqrt{(2\pi)}]\exp\{-[\ln(w)-\mu]^2/2\sigma^2\}, \quad w\varepsilon(0,\infty),\ \mu\varepsilon(-\infty,+\infty),\ \sigma>0 \quad (35)$$

These two distributions have been known as good candidates for presenting distribution of personal income.

In the case of Pareto density function of (34), we can simply derive the Lorenz curve function as follows. Let F(w) denote the Pareto distribution function:

$$F(w)=1-(k/w)^{\theta} \quad (36)$$

with mean equal to,

$$E(w)=\theta^{k}/(\theta-1),\ \ \theta>1 \quad (37)$$

If we find the function y as stated by (31) as a function of x, the Lorenz function will be derived. Now, proceed as follows. Rearrange the terms of (31) as,

$$x(v) = \int_{-\infty}^{v} f(w)dw \quad (38)$$

$$y(x(v)) = [1/E(x)]\int_{-\infty}^{tv} wf(w)dw \quad (39)$$

Substitute Pareto distribution function,

$$x(v) = F(v) = 1-(k/v)^{\theta} \quad (40)$$

$$y(x(v)) = [(\theta-1)/\theta^{k}]\int_{k}^{v} w\theta k^{\theta}w^{-\theta-1}dw \quad (41)$$

or,

$$y(x(v)) = 1-(k/v)^{\theta-1} \quad (42)$$

Now, by solving (40) for "v" and substituting in (42), the Lorenz curve for Pareto distribution is derived as,

$$y = 1-(1-x)^{(\theta-1)/\theta} \quad (43)$$



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As it was shown in the case of Pareto distribution, formula of Lorenz curve is easily obtained. But, if we select the log-normal density function (35), the procedure may not be the same. Because the integral of log-normal function has not been derived yet. In the following pages, the $L_1$ norm smoothing technique will be developed to estimate the parameters of given functional forms (32) and (33) by using the continuous probability density function.

According to (30) and (31) independent and dependent variables of (32) and (33) may be written as,

$$x(v) = \int_0^v f(w)dw \qquad (44)$$

$$y(x(v)) = [1/E(x)] \int_0^v wf(w)dw \qquad (45)$$

Substitute (44) and (45) inside (32) and define random error term u as,

$$[1/E(w)]\int_0^v wf(w)dw = \int_0^v f(w)dw.A^{\int_0^v f(w)dw-1} . e^u \qquad (46)$$

or briefly,

$$y(x)=xA^{x-1}e^u \qquad (47)$$

Similarly for the model (35),

$$[1/E(w)]\int_0^v wf(w)dw=\{\int_0^v f(w)dw\}^B . A^{\int_0^v f(w)dw-1} . e^u \qquad (48)$$

or briefly,

$$y(x)=x^B A^{x-1}e^u \qquad (49)$$

Taking natural logarithm of (47) and (49), gives,

$$\ln y(x)=\ln x + (x-1)\ln A + u \qquad (50)$$

$$\ln y(x)=B.\ln x + (x-1)\ln A + u \qquad (51)$$

With respect to properties of Lorenz curve and probability density function of f(w) and equations (46) to (49), it is obvious that x belongs to the interval [0,1]. Thus the $L_1$ norm objective function for minimizing (50) or (51) is given by,

$$\min: S = \int_0^1 |u| dx \qquad (52)$$

Now, let us deal with $L_1$ norm estimation of "A" of Lorenz curve functional form (32) (redefined by (50)). The corresponding $L_1$ norm objective function will be,

$$\min_A: S = \int_0^1 |\ln y(x) - \ln x - (x-1) \ln A| dx \qquad (53)$$

or,

$$\min_A: S = \int_0^1 |x-1| |[\ln y(x)-\ln x]/(x-1) - \ln A| dx \qquad (54)$$

By a similar technique used by (9), we can rewrite (54) as,

$$\min_A: S = \int_0^t |x-1| \{[\ln y(x)-\ln x]/(x-1)-\ln A\}dx - \int_t^1 |x-1| \{[\ln y(x)-\ln x]/(x-1)-\ln A\}dx \qquad (55)$$

since, $0 \leq x \leq 1$ we have,

$$\min_A: S = - \int_0^t [\ln y(x) - \ln x - (x-1) \ln A]dx + \int_t^1 [\ln y(x) - \ln x - (x-1) \ln A]dx \qquad (56)$$

Differentiate (56) partially with respect to "t" and "A" and equate them to zero;

$$\frac{\delta S}{\delta A} = + \int_0^t [(x-1)/A]dx - \int_t^1 [(x-1)/A]dx = 0 \qquad (57)$$

$$\frac{\delta S}{\delta t} = - 2[\ln y(t) - \ln t - (t-1)\ln A] = 0 \qquad (58)$$



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From equation (57), we have,

$$t = 1 \pm \sqrt{2}/2 \tag{59}$$

Since "t" should belong to the interval [0,1], we accept,

$$t = 1 - \sqrt{2}/2 \tag{60}$$

Substitute (60) in (58), and solve for "A", gives the $L_1$ norm estimation for "A" equal to,

$$A = \left[\frac{1-\sqrt{2}/2}{y(1-\sqrt{2}/2)}\right]^{\sqrt{2}} \tag{61}$$

Now, let us apply this procedure to another Lorenz curve functional form of (33) (redefined by (51)). Rewrite L1 norm objective function (52) for the model (51),

$$\min_{A,B}: S = \int_0^1 \left|\ln y(x) - B \ln x - (x-1) \ln A\right| dx \tag{62}$$

or,

$$\min_{A,B}: S = \int_0^1 |x-1| \left|[\ln y(x)]/(x-1) - (\ln x)/(x-1) - \ln A\right| dx \tag{63}$$

The objective function (63) - by some changing on variables - is similar to (16). Thus, by a similar procedure to those of (17) through (29) we can write "S" as,

$$\min_{A,B}: S = \int_0^{t_1} |x-1| \{[\ln y(x)]/(x-1) - (\ln x)/(x-1) - \ln A\} dx$$

$$- \int_{t_1}^{t_2} |x-1| \{[\ln y(x)]/(x-1) - (\ln x)/(x-1) - \ln A\} dx$$

$$+ \int_{t_1}^{1} |x-1| \{[\ln y(x)]/(x-1) - (\ln x)/(x-1) - \ln A\} dx \tag{64}$$

Since $0 \le x \le 1$, then (64) reduces to,

$$\min_{A,B}: S = -\int_0^{t_1} [\ln y(x) - B \ln x - (x-1) \ln A] dx + \int_{t_1}^{t_2} [\ln y(x) - B \ln x - (x-1) \ln A] dx$$

$$- \int_{t_2}^{1} [\ln y(x) - B \ln x - (x-1) \ln A] dx \tag{65}$$

Differentiate "S" partially with respect to "A", "B", $t_1$ and $t_2$ and equate them to zero,

$$\frac{\delta S}{\delta A} = -\frac{1}{A}\left[\int_0^{t_1} (x-1)dx - \int_{t_1}^{t_2} (x-1)dx + \int_{t_2}^{1} (x-1)dx\right] = 0 \tag{66}$$

$$\frac{\delta S}{\delta B} = \int_0^{t_1} \ln(x)dx - \int_{t_1}^{t_2} \ln(x)dx + \int_{t_2}^{1} \ln(x)dx = 0 \tag{67}$$

$$\frac{\delta S}{\delta t_1} = -2\{\ln[y(t_1)] - B\ln(t_1) - (t_1-1)\ln(A)\} = 0 \tag{68}$$

$$\frac{\delta S}{\delta t_2} = 2\{\ln[y(t_2)] - B\ln(t_2) - (t_2-1)\ln(A)\} = 0 \tag{69}$$

The above system of simultaneous equations can be solved for the unknowns $t_1$, $t_2$, "A" and "B". Equation (66) is reduced to,

$$t_1^2 - t_2^2 - 2(t_1 - t_2) - 1/2 = 0 \tag{70}$$

Equation (67) can be written as,

$$t_1(\ln t_1 - 1) - t_2(\ln t_2 - 1) - 1/2 = 0 \tag{71}$$

Calculate t1 from (70) as,

$$t_1 = 1 \pm \sqrt{\ } q\,(t_2^2 - 2t_2 + 3/2) \tag{72}$$

Since 0st1s1, we accept,


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$$t_1 = 1 - \sqrt{(t_2^2-2t_2+3/2)} \tag{73}$$

Substitute $t_1$ from (73) into (71), and rearrange the terms, gives;

$$\ln \frac{[1-\sqrt{(t_2^2-2t_2+3/2)}]^{[1-\sqrt{(t_2^2-2t_2+3/2)}]}}{t_2^{t2}} + t_2-3/2+\sqrt{(t_2^2-2t_2+3/2)} = 0 \tag{74}$$

The root of equation (74) may be computed by a suitable numerical algorithm. However, it has been computed and rounded for five digits decimal point as,

$$t_2 = 0.40442 \tag{75}$$

Value of t1 is derived by substituting t2 into (73);

$$t_1 = 0.07549 \tag{76}$$

Values of "B" and "A" are computed from (68) and (69) using $t_2$ and $t_1$ given by (75) and (76). Thus,

$$B = \frac{(t_2-1)\ln y(t_1) - (t_1-1)\ln y(t_2)}{(t_2-1)\ln(t_1) - (t_1-1)\ln(t_2)} \tag{77}$$

or,

$$B = -0.84857\ln[y(0.07549)] + 1.31722\ln[y(0.40442)] \tag{78}$$

and,

$$A = [y(0.07549)]^{1.28986}[y(0.40442)]^{-3.68126} \tag{79}$$

Now, let us describe how equation (61) for the model (32) and equations (78) and (79) for the model (33) can be used to estimate the parameters of the Lorenz curve when the probability distribution function is known. In the model (32) we should solve (44) for $x(v)=1-\sqrt{2}/2$. On the other hand, we should find value of "v" such that,

$$x(v) = \int_0^v f(w)dw = 1-\sqrt{2}/2 \tag{80}$$

By substituting this value of "v" into (45), value of $y(1-\sqrt{2}/2)$ is computed. The value $y(1-\sqrt{2}/2)$ is used to compute the parameter "A" given by (61) for model (32).

The procedure for the model (33) is also similar, with the difference that two values of "v" should be computed. Once two different values of "v" are computed as follow,

$$x(v) = \int_0^v f(w)dw = 0.07549 \tag{81}$$

$$x(v) = \int_0^v f(w)dw = 0.40442 \tag{82}$$

Values of "v" are substituted in (45) to find y(0.07549) and y(0.40442). These values of "y" are used to compute the parameters of the model (33) by substituting them into (78) and (79).

The only problem remains is computation of related definite integrals of x(v) defined by (80), (81) and (82) which can be done by appropriate numerical methods such as the enclosed sample computer program coded for MathCAD 11 for a complete example.

## 7. Income distribution in the United States of America

In order to compute the Lorenz curve for the United States, we try to apply the above procedure for both (32) and (33) propositions and using log-normal distribution function assumption. The source of data is "the U.S. economic report of the president to parliament, different years". Median income and disposable personal income per family report by table 1. The amount of mean and median of income were used to derive the log-normal density function parameters $\mu$ and $\delta$. The explained procedure of estimation then applied to the series of data for 1977-2002, and corresponding results are reported in next table 2. The results of Slottje (1989), which are based on quintile data calculations, confirm our finding figures partially. Comparisons show the high compatibility of both procedures. An interesting finding of these calculations is that the distribution of income obeys counter-wise business cycles fluctuations. This finding is a new area for research in the realm of the theory and application of income distribution and business cycles interrelationship.

A sample computer program is also enclosed at the end of these pages.



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Table I.

| Year | Population millions | No. of families millions | Disposable personal income, billions of current $ | Per capita disposable income $ | Per family disposable income $ | Family median Income current $ | Gross domestic product billions of $ | Real gross domestic product billions of chained (2000) $ |
|---|---|---|---|---|---|---|---|---|
| 1977 | 220.3 | 57.2 | 1435.7 | 6,517 | 25,098 | 16009.0 | 2,030.9 | 4,750.5 |
| 1978 | 222.6 | 57.8 | 1608.3 | 7,224 | 27,825 | 17639.9 | 2,294.7 | 5,015.0 |
| 1979 | 225.1 | 59.6 | 1793.5 | 7,967 | 30,091 | 19587.2 | 2,563.3 | 5,173.4 |
| 1980 | 227.7 | 60.3 | 2009.0 | 8,822 | 33,317 | 21023.2 | 2,789.5 | 5,161.7 |
| 1981 | 230.0 | 61.0 | 2246.1 | 9,765 | 36,820 | 22387.8 | 3,128.4 | 5,291.7 |
| 1982 | 232.2 | 61.4 | 2421.2 | 10,426 | 39,432 | 23433.3 | 3,255.0 | 5,189.3 |
| 1983 | 234.3 | 62.0 | 2608.4 | 11,131 | 42,070 | 24673.9 | 3,536.7 | 5,423.8 |
| 1984 | 236.4 | 62.7 | 2912.0 | 12,319 | 46,446 | 26433.1 | 3,933.2 | 5,813.6 |
| 1985 | 238.5 | 63.6 | 3109.3 | 13,037 | 48,890 | 27735.2 | 4,220.3 | 6,053.7 |
| 1986 | 240.7 | 64.5 | 3285.1 | 13,649 | 50,932 | 29458.2 | 4,462.8 | 6,263.6 |
| 1987 | 242.8 | 65.2 | 3458.3 | 14,241 | 53,042 | 30970.2 | 4,739.5 | 6,475.1 |
| 1988 | 245.1 | 65.8 | 3748.7 | 15,297 | 56,971 | 32191.0 | 5,103.8 | 6,742.7 |
| 1989 | 247.4 | 66.1 | 4021.7 | 16,257 | 60,844 | 34213.1 | 5,484.4 | 6,981.4 |
| 1990 | 250.2 | 66.3 | 4285.8 | 17,131 | 64,643 | 35353.3 | 5,803.1 | 7,112.5 |
| 1991 | 253.5 | 67.2 | 4464.3 | 17,609 | 66,435 | 35938.7 | 5,995.9 | 7,100.5 |
| 1992 | 256.9 | 68.2 | 4751.4 | 18,494 | 69,670 | 36573.1 | 6,337.7 | 7,336.6 |
| 1993 | 260.3 | 68.5 | 4911.9 | 18,872 | 71,709 | 36929.5 | 6,657.4 | 7,532.7 |
| 1994 | 263.5 | 69.3 | 5151.8 | 19,555 | 74,341 | 38781.9 | 7,072.2 | 7,835.5 |
| 1995 | 266.6 | 69.6 | 5408.2 | 20,287 | 77,705 | 40610.6 | 7,397.7 | 8,031.7 |
| 1996 | 269.7 | 70.2 | 5688.5 | 21,091 | 81,033 | 42300.2 | 7,816.9 | 8,328.9 |
| 1997 | 273.0 | 70.9 | 5988.8 | 21,940 | 84,467 | 44568.2 | 8,304.3 | 8,703.5 |
| 1998 | 276.2 | 71.6 | 6395.9 | 23,161 | 89,330 | 46736.8 | 8,747.0 | 9,066.9 |
| 1999 | 279.3 | 73.2 | 6695.0 | 23,968 | 91,461 | 48789.3 | 9,268.4 | 9,470.3 |
| 2000 | 282.5 | 73.8 | 7194.0 | 25,467 | 97,478 | 50731.7 | 9,817.0 | 9,817.0 |
| 2001 | 285.6 | 74.3 | 7469.4 | 26,156 | 100,531 | 51407.4 | 10,100.8 | 9,866.6 |
| 2002 | 288.6 | 75.6 | 7857.2 | 27,223 | 103,932 | 51680.0 | 10,480.8 | 10,083.0 |
| 2003 | 290.5 | | 8039.2 | 27,675 | | | 10,735.8 | 10,210.4 |

http://www.gpoaccess.gov/eop/

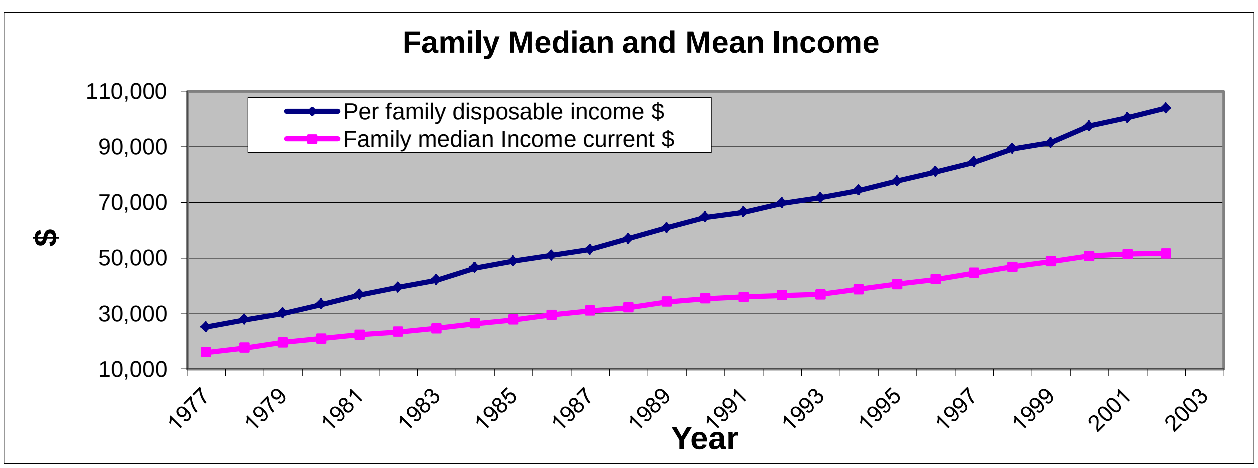




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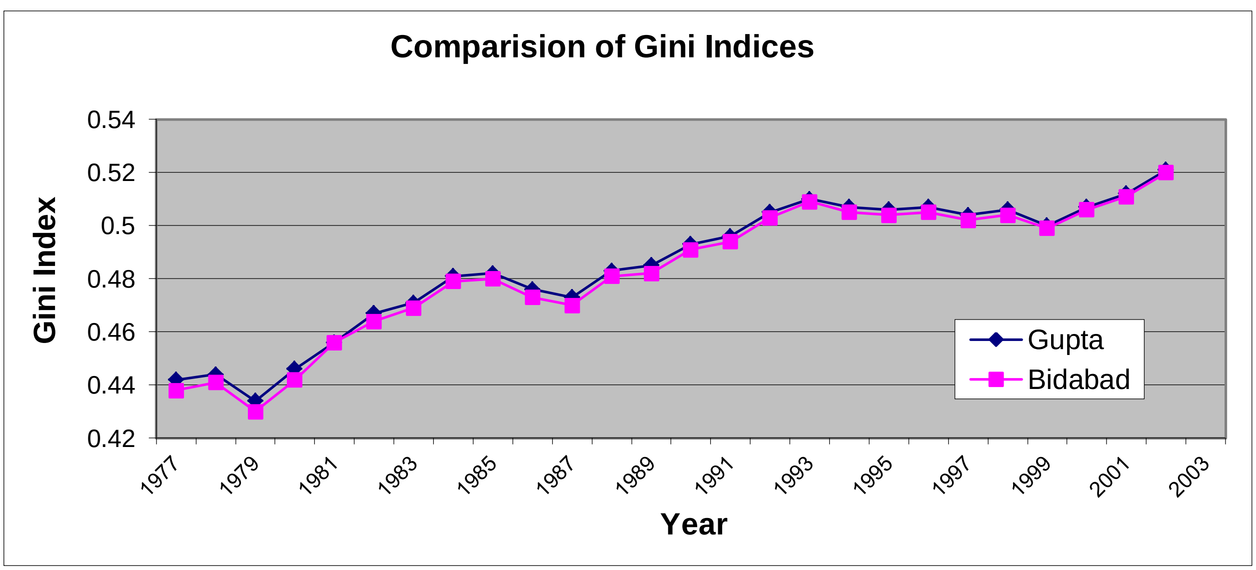
Table 2

| Year | Gupta Model | | | Bidabad model | | | | Slottje figures | |
|---|---|---|---|---|---|---|---|---|---|
| | A | Gini | Kakwani | A | B | Gini | Kakwani | Gini | Kakwani |
| 1977 | 7.938 | 0.442 | 0.172 | 5.798 | 1.214 | 0.438 | 0.170 | 0.426 | 0.109 |
| 1978 | 8.080 | 0.444 | 0.173 | 5.899 | 1.214 | 0.441 | 0.172 | 0.427 | 0.108 |
| 1979 | 7.484 | 0.434 | 0.166 | 5.475 | 1.212 | 0.430 | 0.164 | 0.427 | 0.111 |
| 1980 | 8.189 | 0.446 | 0.175 | 5.978 | 1.215 | 0.442 | 0.173 | 0.428 | 0.112 |
| 1981 | 9.095 | 0.456 | 0.185 | 6.631 | 1.218 | 0.546 | 0.183 | 0.435 | 0.114 |
| 1982 | 9.693 | 0.467 | 0.191 | 7.064 | 1.220 | 0.464 | 0.190 | 0.447 | 0.118 |
| 1983 | 10.051 | 0.471 | 0.194 | 7.324 | 1.221 | 0.469 | 0.193 | 0.447 | 0.120 |
| 1984 | 10.909 | 0.481 | 0.202 | 7.952 | 1.222 | 0.479 | 0.201 | 0.449 | 0.121 |
| 1985 | 11.004 | 0.482 | 0.203 | 8.021 | 1.223 | 0.480 | 0.202 | | |
| 1986 | 10.442 | 0.476 | 0.198 | 7.609 | 1.222 | 0.473 | 0.197 | | |
| 1987 | 10.175 | 0.473 | 0.196 | 7.416 | 1.221 | 0.470 | 0.194 | | |
| 1988 | 11.123 | 0.483 | 0.204 | 8.110 | 1.223 | 0.481 | 0.203 | | |
| 1989 | 11.269 | 0.485 | 0.205 | 8.216 | 1.223 | 0.482 | 0.204 | | |
| 1990 | 12.137 | 0.493 | 0.212 | 8.858 | 1.224 | 0.491 | 0.211 | | |
| 1991 | 12.493 | 0.496 | 0.215 | 9.122 | 1.225 | 0.494 | 0.214 | | |
| 1992 | 13.518 | 0.505 | 0.222 | 9.886 | 1.226 | 0.503 | 0.221 | | |
| 1993 | 14.207 | 0.510 | 0.226 | 10.403 | 1.226 | 0.509 | 0.226 | | |
| 1994 | 13.741 | 0.507 | 0.223 | 10.052 | 1.226 | 0.505 | 0.223 | | |
| 1995 | 13.676 | 0.506 | 0.223 | 10.004 | 1.223 | 0.504 | 0.222 | | |
| 1996 | 13.717 | 0.507 | 0.223 | 10.034 | 1.226 | 0.505 | 0.222 | | |
| 1997 | 13.339 | 0.504 | 0.221 | 9.751 | 1.226 | 0.502 | 0.220 | | |
| 1998 | 13.637 | 0.506 | 0.223 | 9.973 | 1.226 | 0.504 | 0.222 | | |
| 1999 | 12.962 | 0.500 | 0.218 | 9.472 | 1.225 | 0.499 | 0.217 | | |
| 2000 | 13.825 | 0.507 | 0.224 | 10.115 | 1.226 | 0.506 | 0.223 | | |
| 2001 | 14.470 | 0.512 | 0.229 | 10.600 | 1.226 | 0.511 | 0.227 | | |
| *2002* | 15.759 | 0.521 | 0.235 | 11.573 | 1.227 | 0.520 | 0.235 | | |





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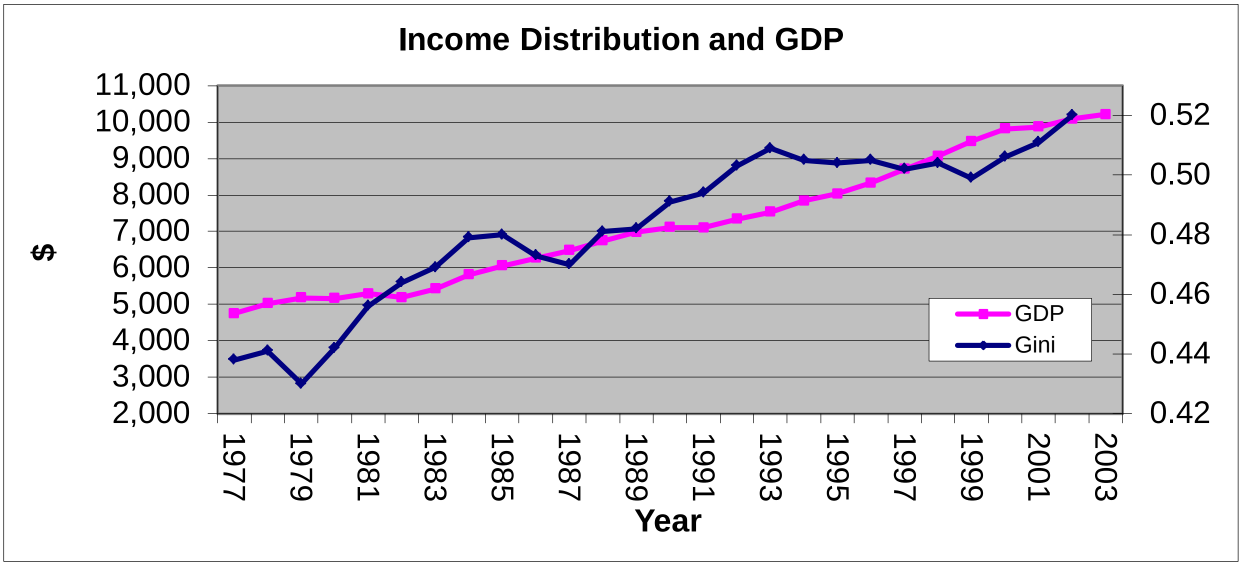
The following graph compares the calculated Gini index with real GDP for the period of 1977-2002.




| Dependent Variable: GINI | | | | |
|---|---|---|---|---|
| Method: Least Squares | | | | |
| Date: 06/23/19 Time: 17:44 | | | | |
| Sample (adjusted): 1979 2002 | | | | |
| Included observations: 24 after adjustments | | | | |
| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
| C | 0.472788 | 0.009894 | 47.78343 | 0.0000 |
| @TREND | 0.002299 | 0.000535 | 4.298753 | 0.0003 |
| GDPGROWTH(-1) | -0.432267 | 0.191001 | -2.263167 | 0.0343 |
| R-squared | 0.521663 | Mean dependent var | | 0.490375 |
| Adjusted R-squared | 0.476107 | S.D. dependent var | | 0.025039 |
| S.E. of regression | 0.018123 | Akaike info criterion | | -5.066782 |
| Sum squared resid | 0.006897 | Schwarz criterion | | -4.919525 |
| Log likelihood | 63.80138 | Hannan-Quinn criter. | | -5.027714 |
| F-statistic | 11.45105 | Durbin-Watson stat | | 2.298751 |
| Prob(F-statistic) | 0.000434 | | | |

As the countercyclical movement of Gini index and GDP is understandable from the above graph, the above simple regression between Gini index and the growth of GDP of USA with one lag also proves this phenomenon. The parameters are meaningful and the t statistics and other statistics are all significant.



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CONTINUOUS L NORM ESTIMATION OF LORENZ CURVE

Bijan BIDABAD

(Using Sample Mean and Median)

Calculations for 2002 USA data

This program has been coded for MathCAD 11

Mean = Sample mean of income distribution: $\text{Mean} := 10393$

Med = Sample median of income distribution: $\text{Med} := 5168$

$$\sigma := \sqrt{2 \cdot \ln\left(\frac{\text{Mean}}{\text{Med}}\right)}$$

Calculation of Log-Normal density function parameters m and s according to sample mean and median

$$\sigma = 1.18209$$

$$\mu := \ln(\text{Med}) \qquad \mu = 10.85283$$

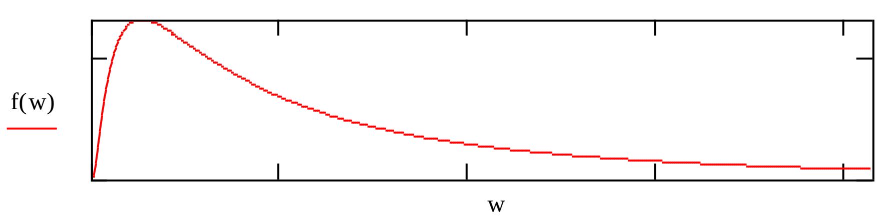
Log-Normal Probability_Density Function

$$f(w) := \left(\frac{1}{w \cdot \sigma \cdot \sqrt{2 \cdot \pi}}\right) \cdot \exp\left[\frac{-(\ln(w) - \mu)^2}{2 \cdot \sigma^2}\right]$$

$$w := 10^{-5}, \frac{\text{Mean}}{200} .. 2 \cdot \text{Mean}$$

Selective range for_Log-Normal plot, values of_increment and upper bound_may be changed

Log-Normal plot




Precision Tolerance level $\quad TOL := 0.0000$

TOL value should be_ changed for more_ accurate solutions,_(less TOL = higher precision)

(45) $$y(v) := \left(\frac{1}{\text{Mean}}\right) \cdot \int_0^v w \cdot f(w)\, dw$$

(44) $$x(v) := \int_{0.00001}^{v} f(w)\, dw$$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Calculation for Gupta model

Initial guess for v. This value should be changed for faster convergence and less iterations

$$v := 20000$$

$$t_0 := 1 - \frac{\sqrt{2}}{2}$$

(60)

Calculating v for (80) $v := \mathrm{root}\left(x(v) - t_0, v\right)$

Calculated v $v = 27136.6437$

y(t )_ 0 $y(v) = 0.04208$ $z_0 := y(v)$

$$A := \left(\frac{t_0}{z_0}\right)^{\sqrt{2}}$$

(61), estimated A: $A = 15.54768$

$$S := \int_0^1 \left|\ln\left(z_0\right) - \ln\left(t_0\right) - \left(t_0 - 1\right)\cdot\ln(A)\right| \, dx$$

(53)

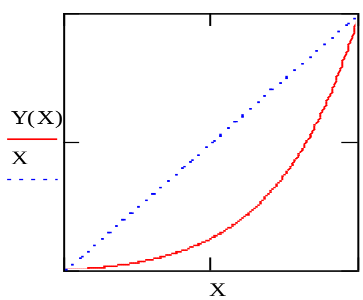
Sum of absolute residuals $S = 0$

Range variable for plotting the Lorenz curves

$$X := 0, 0.005 .. 1$$

$$Y(X) := X \cdot A^{X-1}$$

Gupta Lorenz curve:




Calculation of Gini index

$$\mathrm{Gini} := 1 - 2 \cdot \int_0^1 Y(X) \, dX$$

$$\mathrm{Gini} = 0.51967$$

Calculation of Kakwani length of Lorenz curve

$$\mathrm{Length} := \int_0^1 \sqrt{1 + \left[A^{X-1} \cdot (1 + X \cdot \ln(A))\right]^2} \, dX$$



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Length of Lorenz curve

$$\text{Length} = 1.5515$$

$$\text{Kakwani} := \frac{\text{Length} - \sqrt{2}}{2 - \sqrt{2}}$$

Kakwani index of length $\text{Kakwani} = 0.23437$

Calculation For Bidabad Model

(76) $t_1 := 0.0754$

Initial guess for v. This value should be changed for faster convergence and less iterations

$v := 800($

Calculating v for (81) $v := \text{root}\left(x(v) - t_1, v\right)$

Calculated v $v = 9464.04318$

y(0.07549) $y(v) = 0.00442$ $z_1 := y(v)$

(75) $t_2 := 0.4044$

Initial guess for v. This value should be changed for faster convergence and less iterations

$v := 2700($

Calculatig v for (82) $v := \text{root}\left(x(v) - t_2, v\right)$

Calculated v $v = 38826.2580$

y(0.40442) $y(v) = 0.07722$ $z_2 := y(v)$

(79) $A := \left(z_1\right)^{1.28986} \cdot \left(z_2\right)^{-3.6812}$

(78) $B := -0.84857 \ln\left(z_1\right) + 1.31722 \ln\left(z_2\right)$

Estimated A and B: $A = 11.41481$ $B = 1.22709$

(62) $S := \int_0^1 \left| \ln\left(z_1\right) - B \cdot \ln\left(t_1\right) - \left(t_1 - 1\right) \cdot \ln(A) \right| \, dx$

Sum of absolute residuals $S = 0.00002$

Range variable for plotting the Lorenz curves

$X := 0, 0.005 .. 1$

Bidabad Lorenz curve $Y(X) := X^B \cdot A^{X-1}$



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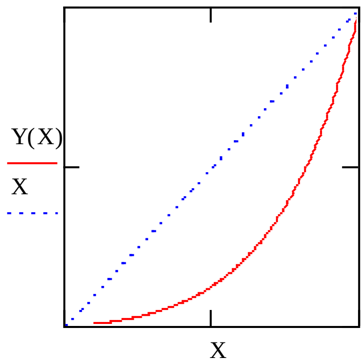
Calculation of Gini Index




$$\text{Gini} := 1 - 2\cdot\int_0^1 Y(X)\,dX$$

$$\text{Gini} = 0.51834$$

Calculation of Kakwani length of Lorenz curve

$$\text{Length} := \int_0^1 \sqrt{1 + \left[A^{X-1}\cdot X^{B-1}\cdot(B + X\cdot\ln(A))\right]^2}\,dX$$

Length of Lorenz curve

$$\text{Length} = 1.55118$$

$$\text{Kakwani} := \frac{\text{Length} - \sqrt{2}}{2 - \sqrt{2}}$$

Kakwani index of length

$$\text{Kakwani} = 0.23381$$

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